Diploma Programme

Mathematics: analysis and approaches HL formula booklet

For use during the course and in the examinations
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HIGHER LEVEL

Contents

Topic 1: Number and algebra – HL

1.2	The nth term of an arithmetic sequence The sum of n terms of an arithmetic sequence	$u_n = u_1 + (n-1)d$ $S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$; $S_n = \frac{n}{2}(u_1 + u_n)$
1.3	The nth term of a geometric sequence The sum of n terms of a finite geometric sequence	$u_n = u_1 r^{n-1}$ $S_n = \frac{u_1(r^n - 1)}{r-1} = \frac{u_1(1-r^n)}{1-r}$, $r \neq 1$
1.8	The sum of an infinite geometric sequence	$S_\infty = \frac{u_1}{1-r}$, $\lvert r \rvert < 1$
1.4	Compound interest	$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$, where FV is the future value, PV is the present value, n is the number of years, k is the number of compounding periods per year, r% is the nominal annual rate of interest
1.5	Exponents and logarithms	$a^x = b \Leftrightarrow x = \log_a b$, where $a > 0, b > 0, a \neq 1$
1.7	Exponents and logarithms Exponential and logarithmic functions	$\log_a xy = \log_a x + \log_a y$ $\log_a \frac{x}{y} = \log_a x - \log_a y$ $\log_a x^m = m \log_a x$ $\log_a x = \frac{\log_b x}{\log_b a}$ $a^x = \mathrm{e}^{x \ln a}$; $\log_a a^x = x = a^{\log_a x}$ where $a, x > 0, a \neq 1$
1.9	Binomial theorem $n \in \mathbb{N}$	$(a+b)^n = a^n + {}^n\mathrm{C}_1\, a^{n-1}b + \ldots + {}^n\mathrm{C}_r\, a^{n-r}b^r + \ldots + b^n$ ${}^n\mathrm{C}_r = \frac{n!}{r!(n-r)!}$

1.10	Combinations	${}^{n}\mathrm{C}_{r} = \dfrac{n!}{r!(n-r)!}$
	Permutations	${}^{n}\mathrm{P}_{r} = \dfrac{n!}{(n-r)!}$
	Extension of binomial theorem, $n \in \mathbb{Q}$	$(a+b)^n = a^n\left(1+n\left(\dfrac{b}{a}\right)+\dfrac{n(n-1)}{2!}\left(\dfrac{b}{a}\right)^2+\ldots\right)$
1.12	Complex numbers	$z = a + b\mathrm{i}$
1.13	Modulus-argument (polar) and exponential (Euler) form	$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta} = r\,\mathrm{cis}\,\theta$
1.14	De Moivre's theorem	$[r(\cos\theta + \mathrm{i}\sin\theta)]^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta) = r^n\mathrm{e}^{\mathrm{i}n\theta} = r^n\,\mathrm{cis}\,n\theta$

Topic 2: Functions – HL

2.1	Equations of a straight line	$y = mx + c$; $ax + by + d = 0$; $y - y_1 = m(x - x_1)$
	Gradient formula	$m = \dfrac{y_2 - y_1}{x_2 - x_1}$
2.6	Axis of symmetry of the graph of a quadratic function	$f(x) = ax^2 + bx + c \Rightarrow$ axis of symmetry is $x = -\dfrac{b}{2a}$
2.7	Solutions of a quadratic equation	$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a},\ a \neq 0$
	Discriminant	$\Delta = b^2 - 4ac$
2.12	Sum and product of the roots of polynomial equations of the form $\sum_{r=0}^{n} a_r x^r = 0$	Sum is $\dfrac{-a_{n-1}}{a_n}$; product is $\dfrac{(-1)^n a_0}{a_n}$

Topic 3: Geometry and trigonometry – HL

Prior learning – HL	
Area of a parallelogram	$A = bh$, where b is the base, h is the height
Area of a triangle	$A = \frac{1}{2}(bh)$, where b is the base, h is the height
Area of a trapezoid	$A = \frac{1}{2}(a+b)h$, where a and b are the parallel sides, h is the height
Area of a circle	$A = \pi r^2$, where r is the radius
Circumference of a circle	$C = 2\pi r$, where r is the radius
Volume of a cuboid	$V = lwh$, where l is the length, w is the width, h is the height
Volume of a cylinder	$V = \pi r^2 h$, where r is the radius, h is the height
Volume of a prism	$V = Ah$, where A is the area of cross-section, h is the height
Area of the curved surface of a cylinder	$A = 2\pi rh$, where r is the radius, h is the height
Distance between two points (x_1, y_1) and (x_2, y_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Coordinates of the midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

3.1	Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
	Coordinates of the midpoint of a line segment with endpoints (x_1, y_1, z_1) and (x_2, y_2, z_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
	Volume of a right-pyramid	$V = \frac{1}{3}Ah$, where A is the area of the base, h is the height

	Volume of a right cone	$V = \frac{1}{3}\pi r^2 h$, where r is the radius, h is the height
	Area of the curved surface of a cone	$A = \pi r l$, where r is the radius, l is the slant height
	Volume of a sphere	$V = \frac{4}{3}\pi r^3$, where r is the radius
	Surface area of a sphere	$A = 4\pi r^2$, where r is the radius
3.2	Sine rule	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
	Cosine rule	$c^2 = a^2 + b^2 - 2ab\cos C$; $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$
	Area of a triangle	$A = \frac{1}{2}ab\sin C$
3.4	Length of an arc	$l = r\theta$, where r is the radius, θ is the angle measured in radians
	Area of a sector	$A = \frac{1}{2}r^2\theta$, where r is the radius, θ is the angle measured in radians
3.5	Identity for $\tan\theta$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
3.6	Pythagorean identity	$\cos^2\theta + \sin^2\theta = 1$
	Double angle identities	$\sin 2\theta = 2\sin\theta\cos\theta$ $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
3.9	Reciprocal trigonometric identities	$\sec\theta = \frac{1}{\cos\theta}$ $\mathrm{cosec}\,\theta = \frac{1}{\sin\theta}$
	Pythagorean identities	$1 + \tan^2\theta = \sec^2\theta$ $1 + \cot^2\theta = \mathrm{cosec}^2\theta$

3.10	Compound angle identities	$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
	Double angle identity for tan	$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$
3.12	Magnitude of a vector	$\lvert \boldsymbol{v} \rvert = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
3.13	Scalar product	$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ $\boldsymbol{v} \cdot \boldsymbol{w} = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \cos\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
	Angle between two vectors	$\cos\theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert}$
3.14	Vector equation of a line	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$
	Parametric form of the equation of a line	$x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$
	Cartesian equations of a line	$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$
3.16	Vector product	$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ $\lvert \boldsymbol{v} \times \boldsymbol{w} \rvert = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \sin\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
	Area of a parallelogram	$A = \lvert \boldsymbol{v} \times \boldsymbol{w} \rvert$ where $\boldsymbol{v}$ and $\boldsymbol{w}$ form two adjacent sides of a parallelogram
3.17	Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$
	Equation of a plane (using the normal vector)	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
	Cartesian equation of a plane	$ax + by + cz = d$

Topic 4: Statistics and probability – HL

4.2	Interquartile range	$\text{IQR} = Q_3 - Q_1$
4.3	Mean, $\bar{x}$, of a set of data	$\bar{x} = \dfrac{\sum_{i=1}^{k} f_i x_i}{n}$, where $n = \sum_{i=1}^{k} f_i$
4.5	Probability of an event A Complementary events	$\text{P}(A) = \dfrac{n(A)}{n(U)}$ $\text{P}(A) + \text{P}(A') = 1$
4.6	Combined events Mutually exclusive events Conditional probability Independent events	$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$ $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B)$ $\text{P}(A \mid B) = \dfrac{\text{P}(A \cap B)}{\text{P}(B)}$ $\text{P}(A \cap B) = \text{P}(A)\,\text{P}(B)$
4.7	Expected value of a discrete random variable X	$\text{E}(X) = \sum_{i=1}^{k} x_i \ \text{P}(X = x_i)$
4.8	Binomial distribution $X \sim \text{B}(n, p)$ Mean Variance	 $\text{E}(X) = np$ $\text{Var}(X) = np(1-p)$
4.12	Standardized normal variable	$z = \dfrac{x - \mu}{\sigma}$
4.13	Bayes' theorem	$\text{P}(B \mid A) = \dfrac{\text{P}(B)\ \text{P}(A \mid B)}{\text{P}(B)\ \text{P}(A \mid B) + \text{P}(B')\ \text{P}(A \mid B')}$ $\text{P}(B_i \mid A) = \dfrac{\text{P}(B_i)\text{P}(A \mid B_i)}{\text{P}(B_1)\text{P}(A \mid B_1) + \text{P}(B_2)\text{P}(A \mid B_2) + \text{P}(B_3)\text{P}(A \mid B_3)}$

4.14	Variance σ^2	$\sigma^2 = \frac{\sum_{i=1}^{k} f_i \left(x_i - \mu\right)^2}{n} = \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2$
	Standard deviation σ	$\sigma = \sqrt{\frac{\sum_{i=1}^{k} f_i \left(x_i - \mu\right)^2}{n}}$
	Linear transformation of a single random variable	$\mathrm{E}\left(aX + b\right) = a\mathrm{E}(X) + b$ $\mathrm{Var}\left(aX + b\right) = a^2\,\mathrm{Var}(X)$
	Expected value of a continuous random variable X	$\mathrm{E}(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\,\mathrm{d}x$
	Variance	$\mathrm{Var}(X) = \mathrm{E}\left[(X-\mu)^2\right] = \mathrm{E}(X^2) - \left[\mathrm{E}(X)\right]^2$
	Variance of a discrete random variable X	$\mathrm{Var}(X) = \sum (x-\mu)^2\ \mathrm{P}(X = x) = \sum x^2\ \mathrm{P}(X = x) - \mu^2$
	Variance of a continuous random variable X	$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x - \mu^2$

Topic 5: Calculus – HL

5.12	Derivative of $f(x)$ from first principles	$y = f(x) \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = \lim\limits_{h \to 0}\left(\dfrac{f(x+h) - f(x)}{h}\right)$
5.3	Derivative of x^n	$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$
5.6	Derivative of $\sin x$	$f(x) = \sin x \Rightarrow f'(x) = \cos x$
	Derivative of $\cos x$	$f(x) = \cos x \Rightarrow f'(x) = -\sin x$
	Derivative of e^x	$f(x) = \mathrm{e}^x \Rightarrow f'(x) = \mathrm{e}^x$
	Derivative of $\ln x$	$f(x) = \ln x \Rightarrow f'(x) = \dfrac{1}{x}$
	Chain rule	$y = g(u)$, where $u = f(x) \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u} \times \dfrac{\mathrm{d}u}{\mathrm{d}x}$
	Product rule	$y = uv \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = u\dfrac{\mathrm{d}v}{\mathrm{d}x} + v\dfrac{\mathrm{d}u}{\mathrm{d}x}$
	Quotient rule	$y = \dfrac{u}{v} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$
5.15	Standard derivatives $\tan x$	$f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$
	$\sec x$	$f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$
	$\operatorname{cosec} x$	$f(x) = \operatorname{cosec} x \Rightarrow f'(x) = -\operatorname{cosec} x \cot x$
	$\cot x$	$f(x) = \cot x \Rightarrow f'(x) = -\operatorname{cosec}^2 x$
	a^x	$f(x) = a^x \Rightarrow f'(x) = a^x(\ln a)$
	$\log_a x$	$f(x) = \log_a x \Rightarrow f'(x) = \dfrac{1}{x \ln a}$
	$\arcsin x$	$f(x) = \arcsin x \Rightarrow f'(x) = \dfrac{1}{\sqrt{1-x^2}}$
	$\arccos x$	$f(x) = \arccos x \Rightarrow f'(x) = -\dfrac{1}{\sqrt{1-x^2}}$
	$\arctan x$	$f(x) = \arctan x \Rightarrow f'(x) = \dfrac{1}{1+x^2}$

5.9	Acceleration	$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$
	Distance travelled from t_1 to t_2	distance $= \int_{t_1}^{t_2} \lvert v(t) \rvert \mathrm{d}t$
	Displacement from t_1 to t_2	displacement $= \int_{t_1}^{t_2} v(t) \mathrm{d}t$
5.5	Integral of x^n	$\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
	Area between a curve $y = f(x)$ and the x-axis, where $f(x) > 0$	$A = \int_a^b y \,\mathrm{d}x$
5.10	Standard integrals	$\int \frac{1}{x} \,\mathrm{d}x = \ln \lvert x \rvert + C$ $\int \sin x \,\mathrm{d}x = -\cos x + C$ $\int \cos x \,\mathrm{d}x = \sin x + C$ $\int \mathrm{e}^x \,\mathrm{d}x = \mathrm{e}^x + C$
5.15	Standard integrals	$\int a^x \,\mathrm{d}x = \frac{1}{\ln a} a^x + C$ $\int \frac{1}{a^2 + x^2} \mathrm{d}x = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$ $\int \frac{1}{\sqrt{a^2 - x^2}} \mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + C,\quad \lvert x \rvert < a$
5.16	Integration by parts	$\int u \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \mathrm{d}x$ or $\int u \,\mathrm{d}v = uv - \int v \,\mathrm{d}u$

5.11	Area of region enclosed by a curve and x-axis	$A = \int_a^b \lvert y \rvert \, \mathrm{d}x$
5.17	Area of region enclosed by a curve and y-axis Volume of revolution about the x or y-axes	$A = \int_a^b \lvert x \rvert \, \mathrm{d}y$ $V = \int_a^b \pi y^2 \, \mathrm{d}x$ or $V = \int_a^b \pi x^2 \, \mathrm{d}y$
5.18	Euler's method Integrating factor for $y' + P(x)y = Q(x)$	$y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$, where h is a constant (step length) $\mathrm{e}^{\int P(x)\mathrm{d}x}$
5.19	Maclaurin series Maclaurin series for special functions	$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$ $\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \ldots$ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$